Toroidal Helical Fields

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Dedicated to Professor Dieter Pfirsch on his 60th Birthday

Using the conventional toroidal coordinate system Laplace's equation for the magnetic scalar potential due to toroidal helical currents is solved. The potential is written as a sum of an infinite series of functions. Each partial sum represents the potential within some accuracy. The effect of the winding law is analysed in the case of small curvature. Approximate magnetic surfaces formed by toroidal helical currents flowing around a standard tokamak chamber are determined. Stability of the plasma column in this system against displacements is discussed.

1. Introduction

For many reasons it has become important to know a correct expression for the field due to a toroidal helical current system. Ohkawa et al. [1] have considered a helically symmetric pinch configuration with pitch reversal (OHTE) for plasma confinement. The pitch reversal is mostly due to external helical windings. Toroidal effects are not taken into account in [1]. According to Sharp et al. [2] some effects of toroidicity can be important for small aspect ratios ($R_0/b \gtrsim 4$) and negligible when $R_0/b \gtrsim 12$. Also, the influence of toroidicity is altered if the helical winding pitch is modulated.

In torsatrons magnetic field configuration is partly due to regularly spaced toroidal helical conductors carrying equal currents in the same direction. Gourdon et al. [3] suggest that choosing conveniently the pitch of the helical winding, one coil could be enough to produce the required stable magnetic field configuration. For these and other reasons analytical expressions for the toroidal helical field due to specified windings have become necessary. Approximate analytical expressions for the toroidal helical field can be obtained considering helically symmetric system bent into a torus ([4], [5], and [6]). The resultant system is neither helical nor axisymmetric. Approximate Laplace's equation, in terms of a local polar coordinate system, for the magnetic scalar potential can be used near the torus. This approximation yields a problem: the boundary conditions for the magnetic field outside the torus become undefined. Sometani et al. [6] in their calculations exclude solutions of the approximate equation outside the winding region that diverge at an infinite distance from the torus axis. The reason does not seem obvious.

In this work, exact Laplace's equation for the magnetic scalar potential is solved using a more natural coordinate system. A particularly simple winding law in this coordinate system is adopted. The potential is written as an infinite series of functions. Each term is of the order of a power of the inverse aspect ratio.

In the case of small curvature system, the first order toroidal effect on the magnetic field is specially analysed. Also, the effect of a non-uniform winding law is considered in this approximation. Both effects can be of the same order of magnitude. The result is compared to the expression obtained by bending a straight system.

Approximate configuration of a plasma in the Brazilian tokamak TBR-1 disturbed by a toroidal helical field [7] is estimated. Stability of the plasma column against displacements is also discussed.

2. Toroidal Helical Currents

In this work, a number of thin conductors wound on a circular torus, carrying currents I in alternating directions is considered.

The toroidal helical winding is characterized by the major and minor radii of the torus R_0 and b,

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respectively, by the number of periods of the helical field in the poloidal and toroidal directions m_0 and n_0 and by a winding law.

The most appropriate coordinate system seems to be the usual toroidal coordinate system defined by ξ, ω and φ (see Fig. 1):

$$r = \frac{R_0' \sinh \xi}{\cosh \xi - \cos \omega} \quad \text{and} \quad z = \frac{R_0' \sin \omega}{\cosh \xi - \cos \omega} \tag{2.1}$$

where r, φ and z are the polar coordinates (Appendix A).

If $\xi = \xi_0$ defines the toroidal surface, then

$$\cosh \xi_0 = R_0/b \quad \text{and} \quad R_0' = R_0 \sqrt{1 - b^2/R_0^2}\,. \tag{2.2}$$

Here, a linear relation between ω and φ is taken as the winding law

$$m_0\, \omega + n_0\, \varphi = \text{const.} \tag{2.3}$$

Also a new toroidal helical coordinate system is defined by ξ, ω and u where $u = m_0\, \omega + n_0\, \varphi$.

The surface current density due to a thin helical toroidal conductor with current I has the components

$$i_\varphi = \frac{m_0\, I}{R_0'} (\cosh \xi_0 - \cos \omega)\, \delta(m_0\, \omega + n_0\, \varphi)$$

and

$$i_\omega = -\frac{n_0\, I}{r}\, \delta(m_0\, \omega + n_0\, \varphi)\,. \tag{2.4}$$

i_φ and i_ω have periodicity $2\pi/m_0$, $2\pi/n_0$ and 2π in the variables ω, φ and u, respectively. Therefore

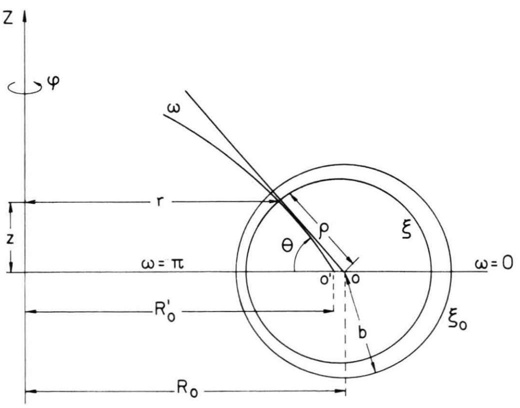

Fig. 1. Coordinate system.

the δ-function can be expanded in Fourier series

$$\delta(u) = \frac{1}{2\pi} \left(1 + 2 \sum_{N=1}^{\infty} \cos Nu\right). \tag{2.5}$$

The resulting current density is

$$i_\varphi = \frac{m_0\, I}{2\pi R_0'} (\cosh \xi_0 - \cos \omega) \left(1 + 2 \sum_{N=1}^{\infty} \cos Nu\right). \tag{2.6}$$

Two simple ways may be considered of producing an n-th harmonic field $\begin{pmatrix} \sin n\,\omega \\ \cos n\,\omega \end{pmatrix}$:

(i) by a pair of conductors with current flow I in opposite directions. The winding must be such that $m_0 = n$. In this case the current density i_φ is given by

$$i_\varphi = \frac{m_0\, I}{R_0'} (\cosh \xi_0 - \cos \omega) \tag{2.7}$$

$$\cdot \left[\delta(m_0\, \omega + n_0\, \varphi) - \delta\left(m_0\left(\omega - \frac{\pi}{m_0}\right) + n_0\, \varphi\right)\right];$$

(ii) by n pairs of conductors with currents $\pm I$ alternately. n must be some multiple of m_0. In this case

$$i_\varphi = \frac{m_0\, I}{R_0'} (\cosh \xi_0 - \cos \omega)$$

$$\cdot \left[\sum_{K=0}^{n-1} \delta\left(m_0\left(\omega - K\frac{2\pi}{n}\right) + n_0\, \varphi\right)\right. \tag{2.8}$$

$$\left. - \sum_{K=0}^{n-1} \delta\left(m_0\left(\omega - K\frac{2\pi}{n} - \frac{\pi}{n}\right) + n_0\, \varphi\right)\right].$$

The resultant current density is

$$i_\varphi = \frac{2 m_0\, I}{\pi R_0'} (\cosh \xi_0 - \cos \omega) \sum_{N=1}^{\infty} \cos Nu\,, \tag{2.9}$$

where $N = 2p + 1$; $p = 0, 1, 2, \ldots, \infty$ in the case (i) and

$$i_\varphi = 2n \frac{m_0\, I}{\pi R_0'} (\cosh \xi_0 - \cos \omega) \sum_{N=1}^{\infty} \cos Nu\,, \tag{2.10}$$

where $N = (2p + 1)\dfrac{n}{m_0}$; $p = 0, 1, 2, \ldots, \infty$ in the case (ii).

i_ω is expressed in terms of $\cos Nu$ in similar fashion. In both cases

$$\cos Nu = \cos\left[(2p + 1)\, n\left(\omega + \frac{n_0}{m_0}\varphi\right)\right]$$

and the field presents an infinite number of harmonics. Near the axis, the predominant contribution to the magnetic field comes from the lowest harmonic term.

3. Boundary Conditions

In toroidal coordinates, boundary conditions on the magnetic field at the surface of the torus are given by

$$B^{e}_{\omega} - B^{i}_{\omega} = -\mu_0 i_\varphi ,$$
$$B^{e}_{\varphi} - B^{i}_{\varphi} = \mu_0 i_\omega , \quad \text{and}$$
$$B^{e}_{\xi} - B^{i}_{\xi} = 0 . \tag{3.1}$$

Quantities in the region outside the helical winding are given a suffix e and those in the region inside the helical winding a suffix i.

Magnetic field may be described in the two regions by the scalar potential $\Phi(\xi, \omega, \varphi)$: $\boldsymbol{B} = \operatorname{grad} \Phi$.

Here, the field of a single harmonic

$$i_\varphi = \frac{m_0 I}{\pi R_0'} (\cosh \xi_0 - \cos \omega) \cos Nu$$

and

$$i_\omega = -\frac{n_0 I}{\pi r} \cos Nu, \quad N \neq 1 \tag{3.2}$$

is deduced.

Boundary conditions for the scalar potential in toroidal helical coordinate system (ξ, ω, u) become

$$\frac{\partial \Phi^e}{\partial \xi} - \frac{\partial \Phi^i}{\partial \xi} = 0 ,$$
$$\frac{\partial \Phi^e}{\partial u} - \frac{\partial \Phi^i}{\partial u} = -\frac{\mu_0 I}{\pi} \cos Nu , \tag{3.3}$$
$$\frac{\partial \Phi^e}{\partial \omega} - \frac{\partial \Phi^i}{\partial \omega} = 0 .$$

The zeroth harmonic

$$i_\varphi = \frac{m_0 I}{2\pi R_0'} (\cosh \xi_0 - \cos \omega)$$

produces zero field inside the winding and

$$i_\omega = -\frac{n_0 I}{2\pi r}$$

produces the toroidal field

$$B^{i}_{\varphi} = \frac{\mu_0}{4\pi} 2 n_0 I .$$

4. The Exact Solution

In toroidal coordinate system, Laplace's equation is

$$\frac{\partial}{\partial \omega} \frac{\sinh \xi}{\cosh \xi - \cos \omega} \frac{\partial \Phi}{\partial \omega} + \frac{\partial}{\partial \xi} \frac{\sinh \xi}{\cosh \xi - \cos \omega} \frac{\partial \Phi}{\partial \xi} + \frac{\partial}{\partial \varphi} \frac{1}{\sinh \xi (\cosh \xi - \cos \omega)} \frac{\partial \Phi}{\partial \varphi} = 0 . \tag{4.1}$$

Introducing a complex function $F(\xi, \omega, \varphi)$ related to Φ by the expression:

$$\Phi = \operatorname{Re} (\cosh \xi - \cos \omega)^{\frac{1}{2}} F(\xi, \omega, \varphi) , \tag{4.2}$$

where Re f means real part of f, we can write for F the equation

$$\frac{\partial^2 F}{\partial \omega^2} + \frac{1}{\sinh \xi} \frac{\partial}{\partial \xi} \sinh \xi \frac{\partial F}{\partial \xi} + \frac{1}{4} F + \frac{1}{\sinh^2 \xi} \frac{\partial^2 F}{\partial \varphi^2} = 0 . \tag{4.3}$$

Boundary conditions (3.3) suggest solutions of the form:

$$F = \sum_\nu F_\nu (\cosh \xi) \, e^{i \nu \omega} e^{i N u} . \tag{4.4}$$

$F_\nu(\cosh \xi)$ is found to satisfy Legendre's associated equation:

$$\frac{d}{d \cosh \xi} (1 - \cosh^2 \xi) \frac{d F_\nu}{d \cosh \xi} + \left\{ \left[(\nu + m_0 N)^2 - \frac{1}{4} \right] - \frac{n_0^2 N^2}{1 - \cosh^2 \xi} \right\} F_\nu = 0 . \tag{4.5}$$

Requiring that in each region the potential be regular, we obtain the following expressions for the interior and exterior fields:

$$F^e_\nu = A^e_\nu P (\cosh \xi)$$

and

$$F^i_\nu = A^i_\nu Q (\cosh \xi) , \tag{4.6}$$

were P and Q are the associated Legendre functions:

$$P \equiv P^{n_0 N}_{m_0 N + \nu - \frac{1}{2}} , \qquad Q \equiv Q^{n_0 N}_{m_0 N + \nu - \frac{1}{2}} .$$

The constants A^{e}_{ν} and A^{i}_{ν} can be determined using the boundary conditions:

$$\sum e^{i\nu\omega}(F^{\mathrm{e}}_{\nu}-F^{\mathrm{i}}_{\nu}) = -\frac{1}{i}\frac{\mu_0 I}{\pi N}(\cosh\xi_0-\cos\omega)^{-\frac{1}{2}} \tag{4.7}$$

and

$$\sum_{\nu} e^{i\nu\omega}\left(\frac{\partial}{\partial\xi}F^{\mathrm{e}}_{\nu}-\frac{\partial}{\partial\xi}F^{\mathrm{i}}_{\nu}\right) = \frac{1}{i}\frac{\mu_0 I}{2\pi N}\frac{\sinh\xi_0}{(\cosh\xi_0-\cos\omega)^{\frac{3}{2}}}$$

deduced from (3.3). On the boundary, $\cosh\xi_0 = R_0/b$.

The function $(\cosh\xi-\cos\omega)^{-\frac{1}{2}}$ is a solution of (4.3) and can be proved to be

$$(\cosh\xi-\cos\omega)^{-\frac{1}{2}} = \frac{\sqrt{2}}{\pi}\sum_{\nu=-\infty}^{+\infty} Q_{\nu-\frac{1}{2}}(\cosh\xi)\,e^{i\nu\omega}. \tag{4.8}$$

Using this expression (4.7) become

$$F^{\mathrm{e}}_{\nu}-F^{\mathrm{i}}_{\nu} = -\frac{\mu_0 I}{i\pi N}\frac{\sqrt{2}}{\pi}Q_{\nu-\frac{1}{2}}(\cosh\xi_0)$$

and (4.9)

$$\frac{1}{\sinh\xi_0}\left(\frac{\partial}{\partial\xi}F^{\mathrm{e}}_{\nu}-\frac{\partial}{\partial\xi}F^{\mathrm{i}}_{\nu}\right)_{\xi_0} = -\frac{\mu_0 I}{i\pi N}\frac{\sqrt{2}}{\pi}Q'_{\nu-\frac{1}{2}}(\cosh\xi_0).$$

The potential inside the winding region can be written

$$\Phi = \sum_{\nu=0}^{\infty}\Phi_{\nu}, \tag{4.10}$$

where

$$\Phi_0 = \frac{\mu_0 I}{2N\pi}\left(\frac{\cosh\xi-\cos\omega}{\cosh\xi_0}\right)^{\frac{1}{2}} f_0(\xi)\sin Nu,$$

$$\Phi_{\nu} = \Phi^{+\nu}+\Phi^{-\nu},$$

$$\Phi^{+\nu} = \frac{\mu_0 I}{2N\pi}\left(\frac{\cosh\xi-\cos\xi}{\cosh\xi_0}\right)^{\frac{1}{2}} \cdot e^{-|\nu|\xi_0}C_{\nu}f_{\nu}(\xi)\sin(Nu+\nu\omega).$$

$C_{\pm\nu}$ are of the order of unity and $f_{\pm\nu}(\xi)\lesssim O(1)$ (Appendix B).

5. Small Curvature Approximation

In the case of small curvature the scalar potential can be represented by

$$\Phi \cong \Phi_0+\Phi_1, \tag{5.1}$$

$$\Phi_0 = \left(\frac{\cosh\xi-\cos\omega}{\cosh\xi_0}\right)^{\frac{1}{2}}\frac{\mu_0 I}{2\pi N}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N+\frac{1}{2}}\sin Nu,$$

$$\Phi_1 = \left(\frac{\cosh\xi-\cos\omega}{\cosh\xi_0}\right)^{\frac{1}{2}}\frac{\mu_0 I}{2\pi N}\frac{b}{4R_0} \cdot\left\{\frac{m_0 N+2}{m_0 N+1}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N+\frac{3}{2}}\sin(Nu+\omega) + \frac{m_0 N}{m_0 N-1}\left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{m_0 N-\frac{1}{2}}\sin(Nu-\omega)\right\}.$$

Terms of the order $\gtrsim O(b/R_0)^2$ are neglected.

From this result, in the case of small curvature, it is also possible to deduce, quite straightforwardly, the scalar potential for a more general winding law of the form

$$m_0(\omega+\delta_0\sin\omega)+n_0\varphi = \text{const.} \tag{5.2}$$

The surface current density due to a single coil becomes

$$i_{\varphi} = \frac{m_0 I}{R'_0}(\cosh\xi_0-\cos\omega)(1+\delta_0\cos\omega) \cdot\delta(m_0(\omega+\delta_0\sin\omega)+n_0\varphi) \tag{5.3}$$

and

$$i_{\omega} = -\frac{n_0 I}{R'_0}(\cosh\xi_0-\cos\omega) \cdot\delta(m_0(\omega+\delta_0\sin\omega)+n_0\varphi)$$

instead of (2.4).

Expression (2.10) must be substituted by

$$i_{\varphi} = 2n\frac{m_0 I}{\pi R'_0}(\cosh\xi_0-\cos\omega)(1+\delta_0\cos\omega) \cdot\sum_{N=1}^{\infty}\cos(Nu+m_0 N\delta_0\sin\omega). \tag{5.4}$$

Using the formula

$$\exp(i\chi\sin\vartheta) = J_0(\chi)+\sum_{K=1}^{\infty}(\pm 1)^K J_K(\chi)\,e^{\pm iK\vartheta}$$

and taking account of only the first two side-bands (5.4) becomes:

$$i_\varphi \cong i_\varphi^{(0)} + i_\varphi^{(\pm 1)}, \tag{5.5}$$

where

$$i_\varphi^{(0)} = 2n\, J_0(m_0 N \delta_0) \frac{m_0 I}{\pi R_0'} (\cosh \xi_0 - \cos \omega) \cdot \cos N(m_0 \omega + n_0 \varphi)$$

and

$$i_\varphi^{(\pm 1)} = 2n \left[\frac{\delta_0}{2} J_0(m_0 N \delta_0) \pm J_1(m_0 N \delta_0) \right] \frac{m_0 I}{\pi R_0'} \cdot (\cosh \xi_0 - \cos \omega) \cos\left(N\left(m_0 \pm \frac{1}{N} \right) \omega + n_0 \varphi \right).$$

J_0 and J_1 are the cylindrical Bessel functions.

The potential due to this system of currents is:

$$\Phi \cong \left(\frac{\cosh \xi - \cos \omega}{\cosh \xi_0} \right)^{\frac{1}{2}} \frac{n \mu_0 I}{\pi N} \left\{ \left(\frac{e^{\xi_0}}{e^{\xi}} \right)^{m_0 N + \frac{1}{2}} J_0(m_0 N \delta_0) \sin Nu + \left(\frac{e^{\xi_0}}{e^{\xi}} \right)^{m_0 N + \frac{3}{2}} \right.$$
$$\cdot \left[\frac{m_0 N + 2}{m_0 N + 1} \frac{b}{4 R_0} J_0(m_0 N \delta_0) + \frac{m_0 N}{m_0 N + 1} \left(\frac{\delta_0}{2} J_0(m_0 N \delta_0) + J_1(m_0 N \delta_0) \right) \right] \sin(Nu + \omega)$$
$$\left. + \left(\frac{e^{\xi_0}}{e^{\xi}} \right)^{m_0 N - \frac{1}{2}} \left[\frac{m_0 N}{m_0 N - 1} \frac{b}{4 R_0} J_0(m_0 N \delta_0) + \frac{m_0 N}{m_0 N - 1} \left[\frac{\delta_0}{2} J_0(m_0 N \delta_0) - J_1(m_0 N \delta_0) \right] \right] \sin(Nu - \omega) \right\}. \tag{5.6}$$

The magnetic field can be obtained by

$$B_\omega = \frac{\cosh \xi - \cos \omega}{R_0'} \frac{\partial \Phi}{\partial \omega} \quad \text{and} \quad B_\xi = \frac{\cosh \xi - \cos \omega}{R_0'} \frac{\partial \Phi}{\partial \xi}. \tag{5.7}$$

If the approximation:

$$e^{-\xi} = e^{-\xi_0} \frac{\varrho}{b} \quad \textit{and} \quad \omega \cong \pi - (\vartheta + \varepsilon \sin \vartheta),$$

where $\varepsilon = \dfrac{\varrho^2 + b^2}{2 R_0 \varrho}$ (Appendix A) is used, near the toroidal surface, the scalar potential can be expressed in terms of local polar coordinates by

$$\Phi \cong \left(\frac{R_0}{r} \right)^{\frac{1}{2}} (-1)^{m_0 N + 1} \frac{n \mu_0 I}{\pi N} \left\{ \left[J_0(m_0 N \delta_0) J_0(m_0 N \varepsilon) \left(\frac{\varrho}{b} \right)^{m_0 N} \sin N(m_0 \vartheta - n_0 \varphi) \right] \right.$$
$$- \left(\frac{\varrho}{b} \right)^{m_0 N + 1} \left\{ \left[\frac{b}{4 R_0} \frac{m_0 N + 2}{m_0 N + 1} J_0(m_0 N \delta_0) + \frac{m_0 N}{m_0 N + 1} \right. \right.$$
$$\left. \cdot \left[\frac{\delta_0}{2} J_0(m_0 N \delta_0) + J_1(m_0 N \delta_0) \right] \right] J_0((m_0 N + 1)\varepsilon) - \frac{b}{\varrho} J_1(m_0 N \varepsilon) \Big\} \sin(N(m_0 \vartheta - n_0 \varphi) + \vartheta)$$
$$- \left(\frac{\varrho}{b} \right)^{m_0 N - 1} \left\{ \left[\frac{b}{4 R_0} \frac{m_0 N}{m_0 N - 1} J_0(m_0 N \delta_0) + \frac{m_0 N}{m_0 N - 1} \left[\frac{\delta_0}{2} J_0(m_0 N \delta_0) - J_1(m_0 N \delta_0) \right] \right] J_0((m_0 N - 1)\varepsilon) \right.$$
$$\left. \left. + \frac{\varrho}{b} J_1(m_0 N \varepsilon) \right\} \sin(N(m_0 \vartheta - n_0 \varphi) - \vartheta) \right\}. \tag{5.8}$$

The magnitudes of the side bands are found to be strongly dependent on the winding law parameter δ_0.

The winding law (5.2) in terms of polar coordinates becomes

$$\vartheta + \left(\frac{b}{R_0} - \delta_0\right) \sin \vartheta - \frac{n_0}{m_0} \vartheta = \text{const.} \tag{5.9}$$

The value of the winding law parameter $\delta_0 = b/R_0$ would describe a linear reltion between ϑ and φ as in Sometani's model.

The use of toroidal coordinate system in order to get a series solution of Laplace's equation has some advantages over the other approach, by bending a straight cylinder into a torus: (i) the boundary conditions for the approximate solutions of approximate Laplace's equation outside the torus are not clear. The problem is complicated by the fact that the contribution to the side band potential by a non-uniform winding law may be of the same order of magnitude as the other toroidal effect. (ii) The expression in terms of local polar coordinates cannot be used if scalelength of the order of b^2/R_0 is considered. So, the position of the magnetic axis, for example, cannot be determined. The expression in terms of toroidal coordinates does not present this problem.

In order to compare both results, some corrections must be made to Sometani's [6] result. The scalar potential inside the winding region should be:

$$\begin{aligned}\Phi_{ni} = {} & \frac{2\mu_0 \alpha l a I}{\pi} K_n'(x_a) I_n(x) \sin n(\vartheta - \alpha S) \\ & + \frac{\mu_0 l a I}{2\pi n R} [x K_n'(x_a) I_n(x) \\ & + \{-x_a K_{n\pm 1}'(x_a) + K_{n\pm 1}(x_a)\} I_{n\pm 1}(x)] \\ & \cdot \sin\{(n \pm 1)\vartheta - \alpha n S\},\end{aligned}$$

where

$$x = \alpha (n^2 - \tfrac{1}{4})^{\frac{1}{2}} \varrho .$$

This expression coincides with (5.8) near the surface if $\delta_0 = b/R_0$ and $m_0 N \delta_0 \ll 1$. Similar correction must be made to the external solution. We must not expect that the corrected Sometani's expression is good outside the winding region because the assumption $\varrho/R_0 \ll 1$ is no longer valid in this region.

La Haye et al. [4] studied the effect of the winding law on the magnetic surface structure of a configuration having a large toroidal plasma current and a set of helical currents. A winding law of the form $\vartheta + \varepsilon \sin \vartheta\, (-n_0/m_0)\, \varphi = \text{const.}$ was considered for different values of the parameter ε. From the results of numerical calculations of the magnetic field using the Biot-Savart law they concluded that the best magnetic surface structure occurs around $\varepsilon \cong 0.18$. The surfaces of constant $|\boldsymbol{B}|$ would be nearly circular and nearly centred about the minor axis of the torus.

The values of the minor and major radii of the system considered are $b = 0.235$ m and $R_0 = 1.24$ m. This would correspond to taking $\delta_0 \cong 0$ in (5.9), that is to say a linear relation between the toroidal angular coordinates.

6. Plasma Configuration in a Tokamak

The equation of average magnetic surfaces which are formed by the $m_0 N$-harmonic of the toroidal helical field and a standard tokamak field is deduced in terms of toroidal coordinates, following Morozov and Solov'ev [5]:

$$\psi_0(\bar{\xi}, \bar{\omega}) - \frac{r^2}{B_{\varphi_0}} \overline{B_\omega \hat{B}_\xi} = \text{const.} \tag{6.1}$$

$\psi_0(\bar{\xi}, \bar{\omega})$ is the flux function of the unperturbed plasma system related to the unperturbed magnetic field by:

$$\boldsymbol{B}_0 = \frac{1}{r} \nabla \psi_0 \, x \, \boldsymbol{e}_\varphi + B_{\varphi_0} \boldsymbol{e}_\varphi . \tag{6.2}$$

The bar and the sign ^ denote the following operations:

$$\begin{aligned}\bar{f}(\xi, \omega) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi, \omega, \varphi)\, d\varphi, \\ \hat{f}(\xi, \omega, \varphi) &= \int_0^{\varphi} \tilde{f}(\xi, \omega, \varphi')\, d\varphi', \\ \tilde{f} &= f - \bar{f};\end{aligned} \tag{6.3}$$

B_ω and B_ξ are due to the toroidal helical currents.

The average magnetic surfaces have nearly circular cross sections shifted inwards by a distance proportional to $(I/I_p)^2$ (I is the helical current and I_p the plasma current).

The distortion in the plasma surface is given by

$$\xi \cong \bar{\xi} + r\frac{\hat{B}_\xi}{B_{\varphi_0}}, \qquad \omega \cong \bar{\omega} + r\frac{\hat{B}_\omega}{B_{\varphi_0}}. \tag{6.4}$$

Using the expression (5.6) the average magnetic surface and the approximate plasma surface is estimated for the brazilian tokamak TBR-1 [7] taking:

$R_0 = 0.3$ m,
b (chamber radius) $= 0.11$ m,
a (plasma radius) $= 0.10$ m,
q (safety factor at plasma surface) $= 3.9$,
$I_p = 9.2$ kA,
$I = 250$ A, and
$n_0 = 1$; $m_0 = 2$; $n = 1$ and $N = 1$.

The results are shown in Figure 2. The inwards shift is very small in this case.

Stability condition against both vertical and horizontal motions of the plasma for non-uniform external field may be written as [8, 9]:

$$0 < n < 1.5,$$

where n is the decay index defined as

$$n = -\frac{R}{\bar{B}_z}\frac{\mathrm{d}}{\mathrm{d}R}\bar{B}_z .$$

$\bar{B}_z$ is the average of B_z taken in the plasma transversal cross section and R is the major radius of the plasma. If B_z is written as $B_{z_0} + B_z^{\mathrm{He}}$, where B_z^{He} is due to the helical windings, and $|B_z^{\mathrm{He}}| \ll |B_{z_0}|$ we have

$$n \cong n_0\left(1 - \frac{\overline{B_z^{\mathrm{He}}}}{\bar{B}_{z_0}}\right) + \Delta n, \tag{6.5}$$

where

$$n_0 = -\frac{R}{\bar{B}_{z_0}}\frac{\mathrm{d}\bar{B}_{z_0}}{\mathrm{d}R}, \qquad \Delta n \cong -\frac{R}{\bar{B}_{z_0}}\frac{\partial\, \overline{B_z^{\mathrm{He}}}}{\partial R},$$

and

$$\frac{\partial \bar{B}_z}{\partial R} \cong \frac{1}{\pi a^2}\int_0^{2\pi} \mathrm{d}\omega\, a\cos\omega\, B_z|_{\varrho=a} .$$

Using the same data, Δn is estimated as:

$$\Delta n \cong -\frac{45\cos n_0 N \varphi}{\bar{B}_{z_0}} . \tag{6.6}$$

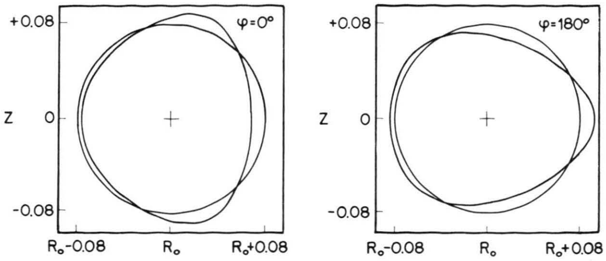

Fig. 2. Shape of the plasma cross section at $\varphi = 0°$ and $\varphi = 180°$. The circular lines represent the cross sections of the toroidal plasma in equilibrium, without perturbation. The other curves show the deformation due to a pair of toroidal helical currents in opposite directions, multiplied by a factor of 5. $R_0 = 0.30$ m, $b = 0.11$ m, $a = 0.08$ m, q (safety factor at plasma surface) $= 3.9$, $I_p = 9.2$ kA, $I = 250$ A, $n_0 = 1$, $m_0 = 2$, $n = 1$ and $N = 1$.

For $\bar{B}_{z_0} \cong -90$ Gauss Δn is $\cong 0.5\cos\varphi$. The decay index is quite sensitive to the helical fields. This means that the stability can be strongly affected by the helical windings. In TBR-1 experiments, without feedback control of the vertical field, the plasma equilibrium was seen to be frequently destroyed by toroidal helical fields [7]: after the helical coil activation, the plasma column moves inwards and touches the vessel. This displacement might be related to the instability against horizontal motions and could be explained by the calculations above.

7. Conclusions

The scalar potential due to toroidal helical currents can be written as an infinite series of functions. Each partial sum represents, everywhere, the solution within an accuracy of the order of $(b/R_0)^s$. The effect of the winding law on the potential is very much similar to first order toroidal effect.

The first order solution can be written in terms of local polar coordinates, far from the axis ($\varrho \gg b^2/R_0$). This expression coincides with corrected Sometani's expression if $m_0 N \delta_0 \ll 1$. In these expressions scale-length of the order of b^2/R_0 is meaningless. Corrected Sometani's expression may not be valid in the external region.

If an standard tokamak field is disturbed by a weak toroidal field ($I \ll I_p$) the equilibrium configuration does not change perceptively. Though, the stability of the plasma column against displacements can be strongly affected.

Appendix A

Relations between toroidal and local polar system:

$$r = R_0 - \varrho \cos \vartheta; \quad z = \varrho \sin \vartheta, \tag{A 1}$$

$$\operatorname{cotan} \omega = \left(1 - \frac{b^2}{R_0^2}\right)^{-\frac{1}{2}} \left[\operatorname{cotan}(\pi - \vartheta) + \frac{\varrho^2 + b^2}{2 R_0 \varrho \sin \vartheta}\right] \tag{A 2}$$

$$\operatorname{cotanh} \xi = \frac{1 - \dfrac{\varrho}{R_0} \cos \vartheta + \dfrac{\varrho^2 - b^2}{2 R_0^2}}{\left(1 - \dfrac{b^2}{R_0^2}\right)^{\frac{1}{2}} \left(1 - \dfrac{\varrho}{R_0} \cos \vartheta\right)}. \tag{A 3}$$

(A 3) may be substituted more conveniently by:

$$e^{-2\xi} = e^{-2\xi_0} \frac{1 - \dfrac{\varrho}{R_0} \cos \vartheta - \dfrac{b}{2 R_0} e^{\xi_0} \left(1 - \dfrac{\varrho^2}{b^2}\right)}{1 - \dfrac{\varrho}{R_0} \cos \vartheta - \dfrac{b}{2 R_0} e^{-\xi_0} \left(1 - \dfrac{\varrho^2}{b^2}\right)}. \tag{A4}$$

If $b/R_0 \ll 1$ (large aspect ratio) and $\varrho/b \gg b/R_0$ (not near the torus axis) we find

$$e^{-\xi} \cong e^{-\xi_0} \frac{\varrho}{b}; \quad \omega \cong \pi - (\vartheta + \varepsilon \sin \vartheta); \qquad \varepsilon = \frac{\varrho^2 + b^2}{2 R_0 \varrho} \tag{A 5}$$

and for the unit vectors

$$\boldsymbol{e}_\xi \cong - \boldsymbol{e}_\varrho, \quad \boldsymbol{e}_\omega \cong - \boldsymbol{e}_\vartheta. \tag{A 6}$$

Appendix B

Legendre functions:

$$P^{n_0 N}_{m_0 N + \nu - \frac{1}{2}}(\cosh \xi) = \frac{\Gamma((m_0 + n_0) N + \nu + \frac{1}{2})}{\Gamma((m_0 - n_0) N + \nu + \frac{1}{2})} \frac{2^{-2 n_0 N}}{\Gamma(1 + n_0 N)} e^{-(m_0 N + \nu + \frac{1}{2}) \xi} (1 - e^{-2\xi})^{n_0 N}$$
$$\cdot F(n_0 N + \tfrac{1}{2}, (m_0 + n_0) N + \nu + \tfrac{1}{2}; 1 + 2 n_0 N; 1 - e^{-2\xi}), \tag{B 1}$$

$$Q^{n_0 N}_{m_0 N + \nu + \frac{1}{2}}(\cosh \xi) = \sqrt{\pi} (-1)^{n_0 N} \frac{\Gamma((m_0 + n_0) N + \nu + \frac{1}{2})}{\Gamma(m_0 N + \nu + 1)} e^{-(m_0 N + \nu + \frac{1}{2}) \xi} (1 - e^{-2\xi})^{n_0 N}$$
$$\cdot F(n_0 N + \tfrac{1}{2}, (m_0 + n_0) N + \nu + \tfrac{1}{2}; m_0 N + \nu + 1; e^{-2\xi}). \tag{B 2}$$

F are hypergeometric functions.

The function (4.6) which satisfies the conditions (4.9) is

$$F^i_\nu = \frac{\mu_0 I}{N \pi} \frac{\sqrt{2}}{\pi} \left(\frac{Q_{\nu - \frac{1}{2}} P - Q'_{\nu - \frac{1}{2}} P}{- W}\right)_{\xi_0} Q(\cosh \xi) \sin(N u + \nu \omega), \tag{B 3}$$

where $- W = P' Q - P Q'$.

Accordingly, the expression for the potential becomes (4.10), where

$$f_\nu(\xi) = \left(\frac{e^{\xi_0}}{e^{\xi}}\right)^{|m_0 N + \nu| + \frac{1}{2}} q_\nu(\xi),$$

$$q_\nu(\xi) = (1 - e^{-2\xi})^{n_0 N} F(n_0 N + \tfrac{1}{2}, (m_0 + n_0) N + \nu + \tfrac{1}{2}, m_0 N + \nu + 1; e^{-2\xi}),$$

$$C_\nu = \frac{2\sqrt{2}}{\pi} (\cosh \xi_0)^{\frac{1}{2}} \left(\frac{Q_{\nu - \frac{1}{2}} P' - Q'_{\nu - \frac{1}{2}} P}{- W}\right)_{\xi_0} E_0 \, e^{-(|m_0 N + \nu| - |\nu| + \frac{1}{2}) \xi_0},$$

$$E_0 = (-1)^{n_0 N} \sqrt{\pi} \frac{\Gamma((m_0 + n_0) N + \nu + \frac{1}{2})}{\Gamma((m_0 - n_0) N + \nu + \frac{1}{2})}.$$

If terms of the order of $(b/R_0)^2$ are neglected, we have

$$C_\nu = \frac{|m_0 N + \nu| + |\nu|}{|m_0 N + \nu|} \frac{\Gamma(|\nu| + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(|\nu| + 1)} .$$

$$f_\nu(\xi) \lesssim q_\nu(\xi) \cong O(1) .$$

Near the surface ($\xi = \xi_0$) each function Φ_ν is of the order of $e^{-|\nu|\xi_0}$.

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